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Normality of $X \times \omega_1$*

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Abstract

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Let ω_1 be the space of countable ordinals. We study the normality of $X \times \omega_1$ under the assumption $t(X) \leq \omega$. We give internal characterizations of X whose product $X \times \omega_1$ with ω_1 is normal. We also discuss the normality of $X \times \omega_1$ for various X—countably paracompact ω_1-compact spaces, GO-spaces, Pol's space and Reed's space.

Keywords: Beslagic-Rudin space, GO-space, normality, Pol's space, product space, Reed's space, ω_1-collectionwise normal, ω_1-compact.

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1. Introduction

Let Y be a nice topological space. Generally speaking, it is difficult to find necessary and sufficient conditions on X to guarantee that $X \times Y$ is normal. For a metric space Y and a compact space Y such conditions are given by Rudin and Starbird [9] and Rudin [8] (see also [5]). For a more special space Y, for example $Y = \omega_1 + 1$, $X \times Y$ is normal if and only if X is ω_1-paracompact and normal (Kunen, see [5, Corollary 3.7]). In this paper we study the normality of $X \times \omega_1$, where ω_1 is the set of countable ordinals with the usual interval topology. According to Nogura

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[6], if X is a compact Hausdorff space, then $X \times \omega_1$ is normal if and only if $t(X) \leqslant \omega$, where $t(X) = \min\{\kappa$: for any subset A of X and any point $x \in \bar{A}$, there exists $B \subset A$ satisfying $|B| \leqslant \kappa$ and $x \in \bar{B}\}$. So it is natural to consider the normality of $X \times \omega_1$ under the assumption $t(X) \leqslant \omega$.

Let $t(X) \leqslant \omega$. In Section 2 we give internal characterizations of X whose product $X \times \omega_1$ is normal. We also show that the normality of $X \times \omega_1$ implies the collectionwise normality and the countable paracompactness of X, but the converse is not true under the assumption $\diamondsuit^{++}$. In Sections 3 and 4 we show the normality of $X \times \omega_1$ when X is an ω_1-compact, countably paracompact, normal space or X is a GO-space. In Section 5 the normality of products for some special spaces—Pol's space and Reed's space—are discussed.

Though almost all definitions and results are easily extended to higher cardinals, for instance $t(X) \leqslant \kappa$ and $X \times \kappa^+$, we limit our discussion to the case $t(X) \leqslant \omega$ and the other factor space is ω_1.

2. The internal characterization

Let $A \subset \omega_1$ and let $\mathscr{H} = \{H_\alpha : \alpha \in A\}$ be a collection of subsets of a space X. We say that $\mathscr{H}$ is *A-continuous* if whenever $\alpha \in A$ and $x \in \overline{\bigcup\{H_\delta : \delta \in (\beta, \alpha) \cap A\}}$ for all $\beta < \alpha$, then $x \in H_\alpha$.

Let $\{Z_\alpha : \alpha < \omega_1\}$ be a collection of subsets of X. We say that $\{Z'_\alpha : \alpha < \omega_1\}$ is the *ω_1-continuous* closure of $\{Z_\alpha : \alpha < \omega_1\}$ if $Z'_\alpha = \bigcap\{\overline{\bigcup\{Z_\delta : \beta < \delta \leqslant \alpha\}} : \beta < \alpha\}$ for all $\alpha < \omega_1$. Note that $\{Z'_\alpha : \alpha < \omega_1\}$ is an ω_1-continuous collection of closed sets.

We say that a collection $\mathscr{H}$ of subsets of X is *closed* or *open* if all members of $\mathscr{H}$ are closed or open, respectively. We say that the collections $\mathscr{H} = \{H_\alpha : \alpha \in A\}$ and $\mathscr{K} = \{K_\alpha : \alpha \in A\}$ with the same index set A are *disjoint* if $H_\alpha \cap K_\alpha = \emptyset$ for all $\alpha \in A$. We say that $\{U_\alpha : \alpha \in A\}$ is an *expansion* of $\{H_\alpha : \alpha \in A\}$ if $H_\alpha \subset U_\alpha$ for all $\alpha \in A$.

Lemma 2.1. *A collection $\mathscr{H} = \{H_\alpha : \alpha < \omega_1\}$ is closed ω_1-continuous if and only if $\bigcup\{H_\alpha \times \{\alpha\} : \alpha < \omega_1\}$ is closed in $X \times \omega_1$.*

Proof. ($\Rightarrow$) Let $\mathscr{H} = \{H_\alpha : \alpha < \omega_1\}$ be an ω_1-continuous collection of closed sets, let $H = \bigcup\{H_\alpha \times \{\alpha\} : \alpha < \omega_1\}$, and suppose $(x, \gamma) \in \bar{H} - H$. Then for any $\beta < \gamma$ and a neighborhood N of x,

$$(N \times (\beta, \gamma]) \cap H \neq \emptyset.$$

Hence $N \cap (\bigcup\{H_\delta : \beta < \delta \leqslant \gamma\}) \neq \emptyset$. This shows $x \in \overline{\bigcup\{H_\delta : \beta < \delta \leqslant \gamma\}}$, so $x \in H_\gamma$, so $(x, \gamma) \in H$.

($\Leftarrow$) Suppose $H = \bigcup\{H_\alpha \times \{\alpha\} : \alpha < \omega_1\}$ is closed in $X \times \omega_1$, and $x \in \overline{\bigcup\{H_\delta : \beta < \delta \leqslant \gamma\}}$ for all $\beta < \gamma$. If $x \notin H_\gamma$, then $(x, \gamma) \notin H$, so there exist a neighborhood N of x and $\beta < \gamma$ such that

$$(N \times (\beta, \gamma]) \cap H = \emptyset.$$

But then $N \cap (\bigcup\{H_\delta : \beta < \delta \leqslant \gamma\}) = \emptyset$. This is a contradiction. $\square$

Theorem 2.2. *The following are equivalent:*

(a) $X \times \omega_1$ *is normal.*

(b) *For each pair of disjoint ω_1-continuous closed collections $\mathcal{H} = \{H_\alpha : \alpha < \omega_1\}$ and $\mathcal{K} = \{K_\alpha : \alpha < \omega_1\}$ there exist disjoint open expansions $\{U_\alpha : \alpha < \omega_1\}$ and $\{V_\alpha : \alpha < \omega_1\}$ of $\mathcal{H}$ and $\mathcal{K}$, respectively, such that $\{X - U_\alpha : \alpha < \omega_1\}$ and $\{X - V_\alpha : \alpha < \omega_1\}$ are ω_1-continuous.*

(c) *For each pair of disjoint ω_1-continuous closed collections $\mathcal{H} = \{H_\alpha : \alpha < \omega_1\}$ and $\mathcal{K} = \{K_\alpha : \alpha < \omega_1\}$ there exists an ω_1-continuous closed expansion $\mathcal{H}' = \{H'_\alpha : \alpha < \omega_1\}$ of $\mathcal{H}$ disjoint from $\mathcal{K}$ such that $H_\alpha \subset \operatorname{int} H'_\alpha$ for all $\alpha < \omega_1$.*

(d) *For each pair of disjoint ω_1-continuous closed collections $\mathcal{H}$ and $\mathcal{K}$ there exists an open expansion $\{U_\alpha : \alpha < \omega_1\}$ of $\mathcal{H}$ whose ω_1-continuous closure is disjoint from $\mathcal{K}$.*

Proof. The equivalence (a)$\Leftrightarrow$(b) is easy, because the conditions on $\{U_\alpha : \alpha < \omega_1\}$ and $\{V_\alpha : \alpha < \omega_1\}$ are equivalent, by Lemma 2.1, to the assertion that $\bigcup \{U_\alpha \times \{\alpha\} : \alpha < \omega_1\}$ and $\bigcup \{V_\alpha \times \{\alpha\} : \alpha < \omega_1\}$ are open in $X \times \omega_1$. Clearly (b)$\Rightarrow$(c), and it is easy to see (c)$\Leftrightarrow$(d). To show (c)$\Rightarrow$(a), we show the following claim:

Claim. *Suppose that the ω_1-continuous closed collections $\mathcal{H} = \{H_\alpha : \alpha < \omega_1\}$ and $\mathcal{K} = \{K_\alpha : \alpha < \omega_1\}$ have disjoint ω_1-continuous closed expansions $\mathcal{H}' = \{H'_\alpha : \alpha < \omega_1\}$ and $\mathcal{K}' = \{K'_\alpha : \alpha < \omega_1\}$ with $H_\alpha \subset \operatorname{int} H'_\alpha$ and $K_\alpha \subset \operatorname{int} K'_\alpha$ for all $\alpha < \omega_1$; then $H = \bigcup \{H_\alpha \times \{\alpha\} : \alpha < \omega_1\}$ and $K = \bigcup \{K_\alpha \times \{\alpha\} : \alpha < \omega_1\}$ can be separated by disjoint open sets in $X \times \omega_1$.*

Proof of Claim. Let $x \in H_\alpha$. Since $x \notin K'_\alpha$, there exists $\beta(x, \alpha) < \alpha$ such that $x \notin \bigcup \{K'_\delta : \beta(x, \alpha) < \delta \leq \alpha\}$. Put $K(x, \alpha) = \bigcup \{K'_\delta : \beta(x, \alpha) < \delta \leq \alpha\}$. Note that $K(x, \alpha)$ is closed. Let $N(x, \alpha) = (\operatorname{int} H'_\alpha) \cap (X - K(x, \alpha))$, and let

$$U = \bigcup \{\bigcup \{N(x, \alpha) \times (\beta(x, \alpha), \alpha] : x \in H_\alpha\} : \alpha < \omega_1\}.$$

For $x \in K_\alpha$, analogously define $\gamma(x, \alpha) < \alpha$, $H(x, \alpha)$, $M(x, \alpha)$ and let

$$V = \bigcup \{\bigcup \{M(x, \alpha) \times (\gamma(x, \alpha), \alpha] : x \in K_\alpha\} : \alpha < \omega_1\}.$$

Then $H \subset U$, $K \subset V$, and U and V are open.

Subclaim. $U \cap V = \emptyset$.

Suppose $U \cap V \neq \emptyset$, then there exist $x \in H_\alpha$, $y \in K_\delta$ such that

$$(N(x, \alpha) \times (\beta(x, \alpha), \alpha]) \cap (M(y, \delta) \times (\gamma(y, \delta), \delta]) \neq \emptyset.$$

Without loss of generality we can assume $\alpha \leq \delta$. Since $N(x, \alpha) \subset H'_\alpha$ and $M(y, \delta) \subset K'_\delta$, $\alpha \neq \delta$. Hence $\alpha < \delta$. This implies that $H'_\alpha \subset H(y, \delta)$ (note that $\gamma(y, \delta) < \alpha$), hence $M(y, \delta) \subset X - H'_\alpha$. But then $M(y, \delta) \cap N(x, \alpha) = \emptyset$, a contradiction, which proves the subclaim, hence the claim.

From the claim, it follows that (c)$\Rightarrow$(a): first use (c) to get an expansion $\mathcal{H}'$ of $\mathcal{H}$ disjoint from $\mathcal{K}$, then again use (c) to get an expansion $\mathcal{K}'$ of $\mathcal{K}$ disjoint from $\mathcal{H}'$, then apply the claim. $\square$

Since $X\times\omega+1$ is a closed subspace of $X\times\omega_1$, the normality of $X\times\omega_1$ implies the countable paracompactness of X. But we will show directly that, by using Theorem 2.2, the normality of $X\times\omega_1$ implies the countable paracompactness and the ω_1-collectionwise normality of X.

Originally, we had assumed $t(X)\leqslant\omega$ in Theorem 2.3. The referee pointed out that the more general result is implicit in Starbird's thesis [12], whereupon we saw how to remove this assumption.

Theorem 2.3. *If $X\times\omega_1$ is normal, then X is countably paracompact and ω_1-collectionwise normal.*

Proof. Since $X\times(\omega+1)$ is a closed subspace of $X\times\omega_1$, the normality of $X\times\omega_1$ implies the countable paracompactness of X. But we will show this fact directly by using the equivalence (a) and (c) in Theorem 2.2.

Countable paracompactness. Suppose that X is not countably paracompact. Then there exists a decreasing sequence $\{H_n : n\in\omega\}$ of closed sets such that $\bigcap\{H_n : n<\omega\}=\emptyset$, but if U_n is open with $H_n\subset U_n$, then $\bigcap\{\bar U_n : n<\omega\}\neq\emptyset$.

Define

$$H_\alpha=\begin{cases}H_n, & \text{if } \alpha=n,\\ \emptyset, & \text{if } \alpha\geqslant\omega,\end{cases}\qquad K_\alpha=\begin{cases}\emptyset, & \text{if } \alpha=n \text{ or } \alpha>\omega,\\ X, & \text{if } \alpha=\omega.\end{cases}$$

Then $\mathscr{H}=\{H_\alpha : \alpha<\omega_1\}$ and $\mathscr{K}=\{K_\alpha : \alpha<\omega_1\}$ are disjoint ω_1-continuous closed collections. If $\mathscr{H}'$ is an ω_1-continuous closed expansion of $\mathscr{H}$ with $H_n\subset \operatorname{int} H'_n$ for all $n\in\omega$, then there exists $x\in\bigcap\{H'_n : n<\omega\}$, so $x\in H'_\omega$. But $K_\omega=X$, so $\mathscr{H}'$ is not disjoint from $\mathscr{K}$, a contradiction.

ω_1-collectionwise normality. Suppose $\mathscr{L}$ is a discrete collection of closed sets which cannot be separated, and $|\mathscr{L}|\leqslant\omega_1$.

Define $\mathscr{H}=\{H_\alpha : \alpha<\omega_1\}$ so that $H_\alpha=\emptyset$ if α is a limit ordinal, and

$$\{H_\alpha : \alpha \text{ is an isolated ordinal}\}=\mathscr{L}.$$

Let

$$K_\alpha=\begin{cases}X, & \text{if } \alpha \text{ is a limit ordinal},\\ \emptyset, & \text{otherwise}.\end{cases}$$

Then $\mathscr{H}$ and $\mathscr{K}$ are disjoint ω_1-continuous closed collections. Suppose $\mathscr{H}'$ is an ω_1-continuous closed expansion of $\mathscr{H}$ disjoint from $\mathscr{K}$, and $H_\alpha\subset\operatorname{int} H'_\alpha$ for all $\alpha<\omega_1$. Since X is normal, we can choose an open set O_α with $H_\alpha\subset O_\alpha\subset H'_\alpha$ and $\bar O_\alpha\cap(\bigcup\{H_\beta : \beta\neq\alpha\})=\emptyset$. Let $U_\alpha=O_\alpha-\bigcup\{O_\beta : \beta<\alpha\}$. The U_α's are disjoint, so for some α, $H_\alpha\not\subset\operatorname{int} U_\alpha$. Hence there exists $x\in H_\alpha$ such that every neighborhood of x meets infinitely many O_β's, $\beta<\alpha$. Let γ_0 be the least ordinal γ such that every neighborhood of x meets infinitely many members of $\{O_\beta : \beta<\gamma\}$. Then $x\in\overline{\bigcup\{H'_\delta : \beta<\delta\leqslant\gamma_0\}}$ for all $\beta<\gamma_0$, so $x\in H'_{\gamma_0}$. Note that γ_0 must be a limit ordinal. But this means $\mathscr{H}'$ is not disjoint from $\mathscr{K}$. □

Theorem 2.4. *Let $t(X) \leq \omega$. Then the following are equivalent:*

(a) *$X \times \omega_1$ is normal.*

(e) *X is countably paracompact and ω_1-collectionwise normal, and for each pair of disjoint ω_1-continuous closed collections $\mathcal{H}$ and $\mathcal{K}$ there exists a closed unbounded set C of ω_1 and an C-continuous closed collection $\{H'_\alpha : \alpha \in C\}$ disjoint from $\mathcal{K}$ with $H_\alpha \subset \operatorname{int} H'_\alpha$ for all $\alpha \in C$.*

(f) *X is countably paracompact, ω_1-collectionwise normal, and for each pair of disjoint ω_1-continuous closed collections $\mathcal{H}$ and $\mathcal{K}$ there exists a closed unbounded set C of ω_1 and open sets $U_\alpha \supset H_\alpha$ for all $\alpha \in C$ such that the C-continuous closure of $\{U_\alpha : \alpha \in C\}$ is disjoint from $\mathcal{K}$.*

Proof. The equivalence of (e) and (f) is easy, and (c) in Theorem 2.2 implies (e) in Theorem 2.4. We will prove that (e) implies (d) in Theorem 2.2.

Let $\mathcal{H}$ and $\mathcal{K}$ be disjoint ω_1-continuous closed collections. Use (e) twice to obtain a closed unbounded set C of ω_1 and disjoint C-continuous closed collections $\{H'_\alpha : \alpha \in C\}$ and $\{K'_\alpha : \alpha \in C\}$ with $H_\alpha \subset \operatorname{int} H'_\alpha$ and $K_\alpha \subset \operatorname{int} K'_\alpha$ for every $\alpha \in C$. Since C is homeomorphic to ω_1, by the claim in the proof of Theorem 2.2 there exist disjoint relatively open sets U, V in $X \times C$ containing $\bigcup\{H_\alpha \times \{\alpha\} : \alpha \in C\}$ and $\bigcup\{K_\alpha \times \{\alpha\} : \alpha \in C\}$, respectively. Let $U_\alpha = U \cap (X \times \{\alpha\})$, $V_\alpha = V \cap (X \times \{\alpha\})$, $\alpha \in C$. Then if $x \in U_\alpha$ (respectively, V_α), there exist a neighborhood N of x and $\beta < \alpha$ such that $N \subset U_\delta$ (respectively, V_δ) for every $\delta \in C \cap (\beta, \alpha]$. Without loss of generality we assume $0 \in C$. For any $\alpha \in C$, let $\alpha' = \inf(C - (\alpha + 1))$, and let

$$L_\alpha = \bigcup\{H_\delta : \alpha < \delta < \alpha'\} - U_{\alpha'}.$$

Claim. *$\{L_\alpha : \alpha \in C\}$ is a locally finite closed collection.*

Proof. Since $L_\alpha = (\bigcup\{H_\delta : \alpha < \delta \leq \alpha'\}) - U_{\alpha'}$, L_α is closed. Suppose $\{L_\alpha : \alpha \in C\}$ is not locally finite at x. Let $\gamma \in C$ be the least ordinal number such that $x \in \overline{\bigcup\{L_\alpha : \alpha < \gamma\}}$. Then for any $\beta < \gamma$, $x \in \overline{\bigcup\{L_\alpha : \beta < \alpha < \gamma\}}$, so $x \in H_\gamma$. Hence $x \in U_\gamma$. Then there exist a neighborhood N of x and $\beta < \gamma$ such that $N \subset U_\delta$ for every $\delta \in C \cap (\beta, \gamma]$. Hence $N \cap L_\delta = \emptyset$ for any δ with $\beta < \delta < \gamma$, which contradicts the minimality of γ. This proves the claim.

Since X is countably paracompact and ω_1-collectionwise normal there exists a locally finite open expansion $\{O_\alpha : \alpha \in C\}$ of $\{L_\alpha : \alpha \in C\}$. Let $L_\alpha \subset O'_\alpha \subset \bar{O}'_\alpha \subset O_\alpha$. For each $\alpha \in C$, by countable paracompactness again, there exist open sets $S_\delta \supset H_\delta$ and $T_\delta \supset K_\delta$, $\alpha < \delta \leq \alpha'$, such that $S_\delta \cap T_\delta = \emptyset$ for all such δ, and such that $\{S_\delta : \alpha < \delta \leq \alpha'\}$ and $\{T_\delta : \alpha < \delta \leq \alpha'\}$ have disjoint $((\alpha'+1) - \alpha)$-continuous closures.

For $\alpha < \delta < \alpha'$, let

$$U_\delta = (S_\delta \cap O_\alpha) \cup [S_\delta \cap (U_{\alpha'} - \bar{O}_{\alpha'})].$$

Then $H_\delta \subset U_\delta$, because if $x \in H_\delta$ but $x \notin O_\alpha$, then $x \in U_{\alpha'}$. If $\delta \in C$, U_δ is already defined. We claim that the ω_1-continuous closure of $\{U_\delta : \delta < \omega_1\}$ is disjoint from $\mathcal{K}$. Suppose not. Then for some $\alpha < \omega_1$ there exists $x \in K_\alpha$ such that for any $\beta < \alpha$,

$x \in \overline{\bigcup\{U_\delta : \beta < \delta < \alpha\}}$. Let α_0 be the least such α. Since $U_\delta \subset S_\delta$ for $\delta \notin C$, clearly $\alpha_0 \in C$; in fact, α_0 is a limit point of C. There exists a neighborhood N of x such that $|\{\delta \in C \cap \alpha_0 : N \cap O_\delta \neq \emptyset\}| < \omega$. Choose $\beta \in C \cap \alpha_0$ such that $N \cap O_\delta = \emptyset$ if $\beta \leq \delta < \alpha_0$. There exist a neighborhood $N' \subset N$ of x and $\beta' \in C \cap \alpha_0$ such that $N' \subset V_\delta$ if $\beta' \leq \delta < \alpha_0$ and $\delta \in C$. Let $\beta'' = \max\{\beta, \beta'\}$. If $\beta'' < \delta < \alpha_0$ and $\delta \in C$, then $N' \subset V_\delta$, so $N' \cap U_\delta = \emptyset$. If $\delta \notin C$, let $\gamma = \max(C \cap \delta)$, then $\gamma \geq \beta'' \geq \beta, \beta'$, so $N' \cap O_\gamma = \emptyset$ and $N' \cap U_\gamma = \emptyset$. Since

$$N' \cap U_\delta = N' \cap \{(S_\delta \cap O_\gamma) \cup [S_\delta \cap (U_{\gamma'} - \bar{O}_{\gamma'})]\} = \emptyset,$$

$N' \cap (\bigcup\{U_\delta : \beta'' < \delta < \alpha_0\}) = \emptyset$. This is a contradiction. □

Assuming $\diamondsuit^{++}$ Beslagic and Rudin [1] construct a space Δ which is collectionwise normal, and every increasing open cover of Δ has a closed shrinking, but there is an open cover having no closed shrinking. We will show that their space Δ gives an example showing that the converse of Theorem 2.3 does not hold. We use notation from [1] freely, to which the reader is referred.

Example 2.5. Let Δ be the space constructed by Beslagic and Rudin assuming $\diamondsuit^{++}$. Let κ in their notation be ω_1. Then Δ is first countable and $\Delta \times \omega_1$ is not normal.

Proof. Recall that $\Delta = \{(\alpha, \beta) \in \omega_1^2 : \beta < \alpha\}$. Let

$$H_\alpha = \{(\beta, \gamma) \in \Delta : \beta \leq \alpha\} \quad (= \Delta_{\alpha+1} \text{ in their notation}),$$

$$K_\alpha = \{(\beta, \gamma) \in \Delta : \gamma \geq \alpha\}.$$

Then $\{H_\alpha : \alpha < \omega_1\}$ is ω_1-continuous since $H_\alpha \subset H_\beta$ for any $\alpha < \beta$, and $\{K_\alpha : \alpha < \omega_1\}$ is ω_1-continuous since $K_\alpha = \bigcap\{K_\beta : \beta < \alpha\}$ and $K_\beta \supset K_\alpha$ whenever $\beta < \alpha$. If $A \subset \omega_1$, let $A_0 = \{(\alpha, 0) : \alpha \in A\}$. The following Claim 1 is immediate from Lemma 2 in [1].

Claim 1. *If $A \subset \omega_1$ is stationary, then $\bar{A}_0 \cap K_\alpha \neq \emptyset$ for every $\alpha < \omega_1$.*

Claim 2. *If $B \subset \omega_1$ is closed unbounded, then there exist α and a sequence $\langle \alpha_n \rangle_{n \in \omega}$ in B converging to α such that the sequence $\langle (\alpha_n, 0) \rangle_{n \in \omega}$ converges to $(\alpha, 0)$.*

Proof of Claim 2. Choose $\alpha \in B$ such that $B \cap \alpha \in \mathscr{C}_\alpha$ and $\alpha \in D$ (i.e., $\mathscr{C}_\alpha$ is closed under finite intersection; note that $\mathscr{C}_\alpha$ and D are from [1]). The sequence $\langle \gamma_n \rangle_{n \in \omega}$ chosen in [1, see bottom two lines of p. 169] is eventually in $B \cap \alpha$. Now it is evident from the definition of $U_n(\alpha, 0)$ on [1, p. 170] that some subsequence of $(\alpha_n, 0)$ converges to $(\alpha, 0)$. The proof of Claim 2 is finished.

Let U_α be an open set with $K_\alpha \subset U_\alpha$ for every $\alpha < \omega_1$. By Claim 1, $C_0^\alpha \subset U_\alpha$ for some closed unbounded set $C^\alpha \subset \omega_1$. Let $C = \Delta_{\alpha < \omega_1} C^\alpha$. Choose $\alpha \in C$ and a sequence $\langle \alpha_n \rangle_{n \in \omega}$ in C converging to α in Claim 2. Then $\alpha_{n+1} \in C^{\alpha_n}$, so $(\alpha_{n+1}, 0) \in U_{\alpha_n}$ for every $n \in \omega$, hence $(\alpha, 0) \in \overline{\bigcup\{U_\delta : \beta < \delta < \alpha\}}$ for every $\beta < \alpha$. Since $(\alpha, 0) \in H_\alpha$, it follows that $\{H_\alpha : \alpha < \omega_1\}$ and $\{K_\alpha : \alpha < \omega_1\}$ cannot be separated. □

3. ω_1-compact spaces

In [4] Kombarov proved that if X is paracompact, $t(X) \leq \omega$ and Y is ω-bounded (a space is said to be *ω-bounded* if the closure of every countable subset is compact), then $X \times Y$ is normal. The careful analysis of his proof shows that if X is ω_1-paracompact and $t(X) \leq \omega$, then $X \times \omega_1$ is normal. We will give another condition on X sufficient to make $X \times \omega_1$ normal.

Lemma 3.1. *Let X be ω_1-compact and $t(X) \leq \omega$. Let $\mathcal{H} = \{H_\alpha : \alpha \in \omega_1\}$ and $\mathcal{K} = \{K_\alpha : \alpha \in \omega_1\}$ be disjoint ω_1-continuous collections. Then there exists an $\alpha < \omega_1$ such that $\bigcup \{H_\beta : \beta > \alpha\}$ and $\bigcup \{K_\beta : \beta > \alpha\}$ are disjoint.*

Proof. Let $H^\alpha = \bigcup \{H_\beta : \beta > \alpha\}$ and $K^\alpha = \bigcup \{K_\beta : \beta > \alpha\}$. Suppose $H^\alpha \cap K^\alpha \neq \emptyset$ for all $\alpha < \omega_1$. We first show the following claim.

Claim. *For each $x \in X$ there exists an $\alpha < \omega_1$ such that $x \notin H^\alpha \cap K^\alpha$.*

Proof of Claim. Assume the contrary. Choose sequences of countable ordinals $\langle \alpha_n \rangle_{n \in \omega}, \langle \beta_n \rangle_{n \in \omega}$ such that

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n < \cdots,$$

$x \in H_{\alpha_n}$, and $x \in K_{\beta_n}$ for all $n \in \omega$.

Let $\alpha = \sup \alpha_n = \sup \beta_n$. Since $\mathcal{H}$ and $\mathcal{K}$ are ω_1-continuous, $x \in H_\alpha \cap K_\alpha$. This is a contradiction.

Using the above claim it is easy to obtain a sequence $\langle x_\beta \rangle_{\beta \in \omega_1}$ of distinct points of X and sequences $\langle \alpha_\beta \rangle_{\beta \in \omega_1}$, $\langle \alpha'_\beta \rangle_{\beta < \omega_1}$, $\langle \alpha''_\beta \rangle_{\beta < \omega_1}$ of countable ordinals such that for each $\beta < \omega_1$:

(1) $\alpha_\gamma \leq \alpha'_\gamma < \alpha_\beta$, $\alpha_\gamma \leq \alpha''_\gamma < \alpha_\beta$ for $\gamma < \beta$,

(2) $\{x_\gamma : \gamma < \beta\} \cap H^{\alpha_\beta} \cap K^{\alpha_\beta} = \emptyset$,

(3) $x_\beta \in H^{\alpha_\beta} \cap K^{\alpha_\beta}$, $x_\beta \in H_{\alpha'_\beta}$, and $x_\beta \in K_{\alpha''_\beta}$.

Note by (3), $\{x_\gamma : \gamma \geq \beta\} \subset H^{\alpha_\beta} \cap K^{\alpha_\beta}$. Since X is ω_1-compact, there exists an accumulation point x of $\langle x_\beta \rangle_{\beta < \omega_1}$. Let γ be the least ordinal number such that $x \in \overline{\{x_\beta : \beta < \gamma\}}$. Such a γ exists because $t(X) \leq \omega$. Let $\delta = \sup\{\alpha_\beta : \beta < \gamma\}$. Then $\delta = \sup\{\alpha'_\beta : \beta < \gamma\} = \sup\{\alpha''_\beta : \beta < \gamma\}$ by (1). Since $x_\beta \in H_{\alpha'_\beta}$ and $x_\beta \in K_{\alpha''_\beta}$ for every $\beta < \gamma$, $x \in H_\delta \cap K_\delta$ by the ω_1-continuity of the collections $\mathcal{H}$ and $\mathcal{K}$. This is a contradiction. □

It is easy to prove the following lemma, so we leave it to the reader.

Lemma 3.2. *Let $\alpha < \omega_1$. Let X be a normal, countably paracompact space. Let $\mathcal{H} = \{H_\beta : \beta < \alpha\}$ and $\mathcal{K} = \{K_\beta : \beta < \alpha\}$ be disjoint α-continuous closed collections. Then $\mathcal{H}$ and $\mathcal{K}$ have disjoint open expansions $\{U_\beta : \beta < \alpha\}$ and $\{V_\beta : \beta < \alpha\}$, respectively such that $\{X - U_\beta : \beta < \alpha\}$ and $\{X - V_\beta : \beta < \alpha\}$ are α-continuous.*

Theorem 3.3. *Let X be a normal, countably paracompact, ω_1-compact space with $t(X) \leqslant \omega$. Then $X \times \omega_1$ is normal.*

Proof. Let $\mathscr{H} = \{H_\alpha : \alpha < \omega_1\}$, $\mathscr{K} = \{K_\alpha : \alpha < \omega_1\}$ be disjoint ω_1-continuous closed collections. By Lemma 3.1 there exists $\alpha < \omega_1$ such that $\bigcup \{H_\beta : \beta > \alpha\}$ and $\bigcup \{K_\beta : \beta > \alpha\}$ are disjoint. Let U and V be disjoint open sets such that $\bigcup \{H_\beta : \beta > \alpha\} \subset U$ and $\bigcup \{K_\beta : \beta > \alpha\} \subset V$. Let $U_\beta = U$, $V_\beta = V$ for $\beta > \alpha$. By Lemma 3.2 disjoint$(\alpha+1)$-continuous closed collections $\{H_\beta : \beta < \alpha+1\}$ and $\{K_\beta : \beta < \alpha+1\}$ have disjoint open expansions $\{U_\beta : \beta < \alpha+1\}$ and $\{V_\beta : \beta < \alpha+1\}$ such that $\{X - U_\beta : \beta < \alpha+1\}$ and $\{X - V_\beta : \beta < \alpha+1\}$ are $(\alpha+1)$-continuous, respectively. Then clearly $\{U_\beta : \beta < \omega_1\}$ and $\{V_\beta : \beta < \omega_1\}$ are disjoint open expansions of $\mathscr{H}$ and $\mathscr{K}$ such that $\{X - U_\beta : \beta < \omega_1\}$ and $\{X - V_\beta : \beta < \omega_1\}$ are ω_1-continuous, respectively. $\square$

As is mentioned in the introduction, if X is compact Hausdorff, then $X \times \omega_1$ is normal if and only if $t(X) \leqslant \omega$ [6]. Chiba [2] gave an example that shows that compactness in this theorem cannot be replaced by paracompactness. The following example shows that it also cannot be replaced by countable compactness (even ω-boundedness).

Example 3.4. Let $X = \{\alpha \in \omega_2 + 1$: the cofinality of α is not $\omega_1\}$. Then X is a normal, countably paracompact, ω-bounded space with $t(X) = \omega_2$.

The proof of the normality of $X \times \omega_1$ is easy, so we leave it to the reader. (Hint: Let H and K be disjoint closed subsets of $X \times \omega_1$. Let $\tilde{H}$ and $\tilde{K}$ be the closure of H and K in the compact space $(\omega_2 + 1) \times (\omega_1 + 1)$, respectively. Show that if $\tilde{H} \cap \tilde{K} \neq \emptyset$, then $H \cap K \neq \emptyset$.)

4. GO-spaces

Let X be a GO-space and u a *right gap* in X, i.e., u is a convex, clopen, initial segment of X with no maximum. Then we say u is a *right Q-gap* [3] if there is a closed discrete set cofinal in u, i.e., there is no cofinal transfinite sequence in u isomorphic to a stationary subset of a regular, uncountable cardinal. A left gap and left Q-gap are defined similarly.

For a space X, $A \subset X \times \omega_1$ and $U \subset X$, let $A_U = A \cap (U \times \omega_1)$. For a GO-space Y and a point $p \in Y$, define $[p, \rightarrow) = \{y \in Y : y \geqslant p\}$. Define $(p, \rightarrow)$, $(\leftarrow, p]$ and $(\leftarrow, p)$ similarly.

Lemma 4.1. *Let $Y = A \cup B$ where A and B are closed in Y. If H and K are disjoint closed subsets of Y such that $\{H \cap A, K \cap A\}$ and $\{H \cap B, K \cap B\}$ have disjoint open expansions in A and B respectively, then $\{H, K\}$ has a disjoint open expansion in Y.*

Proof. By the hypothesis, there exist open sets U_A, U_B, V_A and V_B such that $H \cap A \subset U_A$, $H \cap B \subset U_B$, $K \cap A \subset V_A$, $K \cap B \subset V_B$, $U_A \cap V_A \cap A = \emptyset$ and $U_B \cap V_B \cap B = \emptyset$. Then

$$U = (U_A - B) \cup (U_B - A) \cup (U_A \cap U_B),$$

$$V = (V_A - B) \cup (V_B - A) \cup (V_A \cap V_B)$$

are disjoint open sets containing H and K, respectively. $\square$

Lemma 4.2. *Let X be a first countable GO-space with the first point y, and let H and K be disjoint closed subsets of $X \times \omega_1$. If $\{H_{[y,z]}, K_{[y,z]}\}$ has a disjoint open expansion in $[y, z] \times \omega_1$ for all $z \in X$, then $\{H, K\}$ has a disjoint open expansion in $X \times \omega_1$.*

Proof. Assume X has a right end gap u since otherwise the proof is trivial.

Case 1. u is a Q-gap. If there is a connected set C cofinal in X, let $z \in C$. Then $[z, \rightarrow)$ is Lindelöf, and by the assumption $\{H_{[y,z]}, K_{[y,z]}\}$ has a disjoint open expansion in $[y, z] \times \omega_1$. By Theorem 3.3 $[z, \rightarrow) \times \omega_1$ is normal. (Note that every Lindelöf space is ω_1-compact.) So $\{H, K\}$ has a disjoint open expansion. If no connected set is cofinal in X, then $X = \bigcup \mathcal{D}$, where $\mathcal{D}$ is a pairwise disjoint collection of convex clopen sets none of which is cofinal in X. By the assumption $\{H_D, K_D\}$ has a disjoint open expansion in $D \times \omega_1$ for all $D \in \mathcal{D}$. Hence $\{H, K\}$ has a disjoint open expansion in $X \times \omega_1$.

Case 2. u is a non-Q-gap. Hence there is a copy of a stationary set cofinal in X.

Let r be the right end gap of ω_1. At most one of H and K has "$\langle u, r\rangle$ in its closure". That is, if H has nonempty intersection with each $(p, \rightarrow) \times (\alpha, \rightarrow)$, then K does not. To see this consider two subcases.

Subcase 1. X has cofinality ω_1. Let $\{p_\alpha : \alpha < \omega_1\}$ be a cofinal sequence in X isomorphic to a stationary subset of ω_1, and suppose $(p_\alpha, \rightarrow) \times (\alpha, \rightarrow)$ has non-empty intersection with both H and K for each $\alpha < \omega_1$. Choose

$$\langle a_\alpha, \alpha'\rangle \in H \cap (p_\alpha, \rightarrow) \times (\alpha, \rightarrow),$$

$$\langle b_\alpha, \alpha''\rangle \in K \cap (p_\alpha, \rightarrow) \times (\alpha, \rightarrow),$$

for each $\alpha < \omega_1$ such that $\alpha' < \alpha'' < (\alpha+1)'$ and $a_\alpha < b_\alpha < a_{\alpha+1}$. Since $\{p_\alpha : \alpha < \omega_1\}$ is stationary, there exist limit ordinals α_1 and $\beta < \omega_1$ such that $\langle a_\alpha\rangle_{\alpha<\beta}$ and $\langle b_\alpha\rangle_{\alpha<\beta}$ converge to p_{α_1}. Let $\alpha_0 = \sup\{\alpha' : \alpha < \beta\}$. Since H and K are closed, $\langle p_{\alpha_1}, \alpha_0\rangle \in H \cap K$. This is a contradiction.

Subcase 2. X has cofinality $\lambda > \omega_1$. Let

$$H' = \{\alpha < \omega_1 : H \cap (X \times \{\alpha\}) \text{ is cofinal in } X \times \{\alpha\}\},$$

$$K' = \{\alpha < \omega_1 : K \cap (X \times \{\alpha\}) \text{ is cofinal in } X \times \{\alpha\}\}.$$

Clearly $H' \cap K' = \emptyset$. We first show that H' and K' are closed. Let $\alpha \in \bar{H}'$, and let $\langle \alpha_n\rangle_{n<\omega}$ be a sequence in H' converging to α. For each n let $H_n = \{p \in X : \langle p, \alpha_n\rangle \in H\}$. Then H_n is a closed unbounded subset of X. Hence $\bigcap\{H_n : n < \omega\}$ is closed unbounded in X. So $\langle p, \alpha\rangle \in H$ for all $p \in \bigcap\{H_n : n < \omega\}$, i.e. $\alpha \in H'$. Thus H' is closed. K' is similarly shown to be closed. Therefore there exists $\beta<\omega_1$ such that either H' or K' is contained in $[0,\beta]$, say K'. Since λ is a regular ordinal greater than ω_1, there exists $q\in X$ greater than all p, where $\langle p,\alpha\rangle\in K-K'\times[0,\beta]$ for some $\alpha<\omega_1$. So $(q,\rightarrow)\times(\beta,\rightarrow)$ misses K. In either subcase there exist $p\in X$ and $\alpha<\omega_1$ such that $[p,\rightarrow)\times[\alpha,\rightarrow)$ is disjoint from either H or K. Since $[0,\alpha]$ is compact metric, $X\times[0,\alpha]$ is normal. By the assumption $H_{[y,p]}$ and $K_{[y,p]}$ can be separated in $[y,p]\times\omega_1$. Hence by Lemma 4.1 since $(X\times[0,\alpha])\cup([y,p]\times\omega_1)=X\times\omega_1-((p,\rightarrow)\times(\alpha,\rightarrow))$, $\{H,K\}$ has a disjoint open expansion in $X\times\omega_1$. □

Theorem 4.3. *Let X be a GO-space with $t(X)\leqslant\omega$. Then $X\times\omega_1$ is normal.*

Proof. Note that if X is a GO-space with $t(X)\leqslant\omega$, then X is first countable. Let $x\in X$, and let H and K be disjoint closed subsets of $X\times\omega_1$. On each convex subset V of X containing x consider the following property (∗):

$$\begin{array}{l}\{H_{[p,q]},K_{[p,q]}\}\text{ has a disjoint open expansion in }[p,q]\times\omega_1\\ \text{for all }p\leqslant q\text{ in }V.\end{array}\tag{$*$}$$

Let $\mathcal{U}_x=\{[a,b]\subset X: x\in[a,b],\ \{H_{[a,b]},K_{[a,b]}\}$ has a disjoint open expansion in $[a,b]\times\omega_1\}$, and let $U_x=\bigcup\mathcal{U}_x$. For $p\leqslant q$, $p,q\in U_x$, there exist $I,J\in\mathcal{U}_x$ such that $p\in I$ and $q\in J$. Since $x\in I\cap J$, $I\cup J\in\mathcal{U}_x$ by Lemma 4.1. Hence since the closed set $[p,q]\subset I\cup J$, $\{H_{[p,q]},K_{[p,q]}\}$ has a disjoint open expansion in $[p,q]\times\omega_1$, i.e. U_x satisfies (∗). Clearly since $x\in\bigcap\mathcal{U}_x$, U_x is the largest convex subset of X satisfying (∗). By Lemma 4.2 $\{H_{U_x},K_{U_x}\}$ has a disjoint open expansion in $U_x\times\omega_1$.

We now show that U_x is a neighborhood of x in X. There exists $\alpha<\omega_1$ such that $\{x\}\times[\alpha,\rightarrow)$ is disjoint from either H or K, say H. Then there exists a convex open neighborhood C of x in X such that $H\cap(C\times[\alpha,\rightarrow))=\emptyset$. To see this let $\{C_n: n<\omega\}$ be a decreasing open neighborhood base of x, and suppose there exists $\langle c_n,\alpha_n\rangle\in H\cap(C_n\times[\alpha,\rightarrow))$ for each n. Then $\langle x,\alpha'\rangle\in\overline{\{\langle c_n,\alpha_n\rangle: n<\omega\}}\subset H$ for some $\alpha'\geqslant\alpha$. This is a contradiction. So there exist $p,q\in C$ such that $p\leqslant x\leqslant q$ and $[p,q]$ is a neighborhood of x. Now $H\cap([p,q]\times[\alpha,\rightarrow))=\emptyset$, and $[p,q]\times[0,\alpha]$ is normal. So $\{H_{[p,q]},K_{[p,q]}\}$ has a disjoint open expansion in $[p,q]\times\omega_1$. Hence $[p,q]\subset U_x$, and so U_x is a neighborhood of x.

Now we show U_x is clopen in X. Let $y\in\bar{U}_x$. Then $U_x\cap U_y\neq\emptyset$, and so $U_x\cup U_y$ is convex and satisfies (∗). Hence $U_y\subset U_x$ (in fact $U_y=U_x$), and so U_x is open.

(Now unfix x.) Choose $M\subset X$ such that $U_x\neq U_{x'}$, for all distinct $x,x'\in M$, and $X=\bigcup\{U_x: x\in M\}$. Hence $\{U_x: x\in M\}$ is a clopen partition of X, and $\{H_{U_x},K_{U_x}\}$ can be separated in $U_x\times\omega_1$ for each $x\in M$. So $\{H,K\}$ has a disjoint open expansion in $X\times\omega_1$. Hence $X\times\omega_1$ is normal. □

5. Pol and Reed spaces

Let $D(\omega_1)$ be the space of all ordinals less than ω_1 with the discrete topology. Let $X=D(\omega_1)^\omega$ with Pol's topology, i.e. the topology on X is generated by the metric topology together with sets of the form $X_\alpha = \{f \in X : \sup f \leq \alpha\}$ for every $\alpha < \omega_1$. Then X is a locally second countable (hence $t(X) \leq \omega$), perfectly normal, collectionwise normal, nonparacompact space [10]. We show the following:

Proposition 5.1. *$X \times \omega_1$ is normal.*

Proof. Let H_0 and H_1 be disjoint closed subsets of $X \times \omega_1$. For $\sigma \in \omega_1^n$, let $[\sigma] = \{f \in X : f | n = \sigma\}$, where $f|n$ is the restriction of f to n. Let $H_i^\sigma = H_i \cap ([\sigma] \times \omega_1)$, for $i < 2$. Let $\pi : X \times \omega_1 \to \omega_1$ be the projection.

Claim 1. *Let $O = \bigcup \{[\sigma] : \pi(H_i^\sigma)$ is bounded in ω_1 for some $i < 2\}$. Then the set*

$$S = \{\alpha \in \omega_1 : \text{there is } f \text{ in } X - O \text{ with } \sup f = \alpha\}$$

is nonstationary in ω_1.

Proof of Claim 1. Suppose S is stationary. For each $\alpha \in S$, pick $f_\alpha \in X - O$ with $\sup f_\alpha = \alpha$. Since either $(f_\alpha, \alpha) \notin H_0$ or $(f_\alpha, \alpha) \notin H_1$, there exist $i_\alpha < 2$, $\beta(\alpha) < \alpha$, and a neighborhood $N_\alpha = [f_\alpha | n_\alpha] \cap X_\alpha$ of f_α such that

$$(N_\alpha \times (\beta(\alpha), \alpha]) \cap H_{i_\alpha} = \emptyset.$$

By the pressing-down lemma there exist $i_0 < 2$, $\beta < \omega_1$, $n < \omega$, $\sigma \in \omega_1^n$, and a stationary set $S' \subset S$ such that $i_\alpha = i_0$, $\beta(\alpha) = \beta$, $n_\alpha = n$, and $f_\alpha | n_\alpha = \sigma$ for all $\alpha \in S'$. Then it is easy to check that

$$([\sigma] \times (\beta, \omega_1)) \cap H_{i_0} = \emptyset,$$

so $\pi(H_{i_0}^\sigma)$ is bounded. Then $f_\alpha \in [\sigma] \subset O$ for any $\alpha \in S'$. This is a contradiction.

Claim 2. *The sets $\{H_i \cap (O \times \omega_1)\}$, $i < 2$, can be separated by disjoint open sets.*

Proof of Claim 2. There is a disjoint cover of O by clopen sets $[\sigma]$ such that $\pi(H_{i_\sigma}^\sigma)$ is bounded, where $i_\sigma < 2$. So it suffices to show that H_0^σ and H_1^σ can be separated for such σ. Let $\pi(H_{i_\sigma}^\sigma) \subset [0, \alpha]$. By the normality of $[\sigma] \times [0, \alpha]$, there is an open set $U \subset [\sigma] \times [0, \alpha]$ such that $H_{i_\sigma}^\sigma \subset U$ and

$$(\mathrm{cl}_{[\sigma] \times [0,\alpha]} U) \cap (H_{|i_\sigma - 1|}^\sigma \cap ([\sigma] \times [0, \alpha])) = \emptyset.$$

But $\mathrm{cl}_{[\sigma] \times [0,\alpha]} U = \bar{U}$, so $\bar{U} \cap H_{|i_\sigma - 1|}^\sigma = \emptyset$, and so H_0^σ and H_1^σ can be separated. This proves Claim 2.

Let C be a closed unbounded subset of ω_1 such that $C \cap S = \emptyset$, where S is as in Claim 1. For $\alpha \in C$, let $\alpha' = \sup(\alpha \cap C)$, and let

$$F_\alpha = \{f : \alpha' \leq \sup f \leq \alpha\} - O.$$

It is easy to check that $\{F_\alpha : \alpha \in C\}$ is a closed discrete collection of closed separable metrizable subsets of X. By collectionwise normality of X, there exists a discrete open expansion $\{G_\alpha : \alpha \in C\}$ of $\{F_\alpha : \alpha \in C\}$. Let $G = \bigcup \{G_\alpha : \alpha \in C\}$. Note that $X = G \cup O$.

Claim 3. *$\{H_i \cap (G \times \omega_1): i<2\}$ can be separated.*

Proof of Claim 3. It suffices to show that $\{H_i \cap (\bar{G}_\alpha \times \omega_1): i<2\}$ can be separated in $\bar{G}_\alpha \times \omega_1$ for each $\alpha \in C$. We have $\bar{G}_\alpha \cap (X-O) = F_\alpha \subset X_\alpha$, and X_α is closed in X. By Theorem 3.3 the restriction of H_0 and H_1 to $X_\alpha \times \omega_1$ can be separated, and the restrictions of H_0 and H_1 to $(G_\alpha - X_\alpha) \times \omega_1$ can be separated by Claim 2. Claim 3 easily follows.

To complete the proof of the proposition, let $\{O', G'\}$ be a shrinking of $\{O, G\}$, i.e., $\overline{O'} \subset O$, $\overline{G'} \subset G$ and $O' \cup G' = X$. Let $\{O_0, O_1\}$ and $\{G_0, G_1\}$ be separation of $\{H_0 \cap (O \times \omega_1), H_1 \cap (O \times \omega_1)\}$ and $\{H_0 \cap (G \times \omega_1), H_1 \cap (G \times \omega_1)\}$, respectively. Let $O_0' = O_0 \cap (O' \times \omega_1)$, $G_0' = G_0 \cap (G_1' \times \omega_1)$. It is easy to check that

$$H_0 \subset O_0' \cup G_0' \quad \text{and} \quad (\bar{O}_0' \cup \bar{G}_0') \cap H_1 = \emptyset.$$

The proof of Proposition 5.1 is completed. □

Let $X = \{x_\alpha : \alpha < \omega_1\}$ be a set of reals. We say that X is a Reed space if the topology on X is the intersection topology with respect to the inherited real line topology on X and the order topology on X of type ω_1. Assuming the continuum hypothesis, Reed [7] constructed a collectionwise normal Reed space. Also by Theorem 2 in [7] and Theorem 1 in [5], in a model of MA+$\neg$CH there is no normal Reed space.

Proposition 5.2. *Let X be a normal Reed space. Then $X \times \omega_1$ is normal.*

Proof. By Kunen's lemma [5], there exists a closed unbounded set C of X such that C is ω_1-compact. Let $\{H_\alpha : \alpha < \omega_1\}$ and $\{K_\alpha : \alpha < \omega_1\}$ be disjoint ω_1-continuous closed collections. Let $H^\alpha = \bigcup\{H_\beta : \beta \geqslant \alpha\}$ and $K^\alpha = \bigcup\{K_\beta : \beta \geqslant \alpha\}$. Then by Lemma 3.1, there exists an $\alpha < \omega_1$ such that $(H^\alpha \cap C) \cap (K^\alpha \cap C) = \emptyset$. First we show that there exist disjoint open sets U and V in X such that $H^\alpha \cap C \subset U$, $K^\alpha \cap C \subset V$, $U \cap K^\alpha = \emptyset$ and $V \cap H^\alpha = \emptyset$. Let U', V' be open sets in X such that $H^\alpha \cap C \subset U' \subset \overline{U'}$, $K^\alpha \cap C \subset V' \subset \overline{V'}$, $\overline{U'} \cap \overline{V'} = \emptyset$. Then $\overline{U'} \cap K^\alpha$ and $H^\alpha \cap C$ are disjoint closed sets. Let $U \subset U'$ be an open set such that

$$H^\alpha \cap C \subset U \subset \bar{U} \subset X - (\overline{U'} \cap K^\alpha).$$

Then U is as required. Similarly we can get an open set V disjoint from U. Since $[(H^\alpha \cup K^\alpha) - (U \cup V)] \cap C = \emptyset$, we can choose an open set W in X such that

$$(H^\alpha \cup K^\alpha) - (U \cup V) \subset W \subset \bar{W} \subset X - C.$$

Then W is metrizable (note that W has a σ-discrete base), and $(H^\alpha \cup K^\alpha) - (U \cup V)$ is a closed subset of W. Hence, by the introduction of Section 3 and Theorem 2.2, the $(\omega_1 - (\alpha+1))$-continuous closed collections $\{H_\beta - (U \cup V): \alpha < \beta < \omega_1\}$ and $\{K_\beta - (U \cup V): \alpha < \beta < \omega_1\}$ have disjoint open expansions $\{O_\beta : \alpha < \beta < \omega_1\}$, $\{P_\beta : \alpha < \beta < \omega_1\}$ with $(\omega_1 - (\alpha+1))$-continuous complements. Let $U_\beta = U \cup O_\beta$,

$V_\beta = V \cup P_\beta$ for $\beta > \alpha$. On the other hand, $\{H_\beta : \beta < \alpha\}$ and $\{K_\beta : \beta < \alpha\}$ have disjoint open expansions $\{U_\beta : \beta < \alpha\}$, $\{V_\beta : \beta < \alpha\}$ with α-continuous complements by Lemma 3.2. Then $\{U_\beta : \beta < \omega_1\}$, $\{V_\beta : \beta < \omega_1\}$ are disjoint open expansions of $\{H_\beta : \beta < \omega_1\}$, $\{K_\beta : \beta < \omega_1\}$, respectively. □

By combining the above proposition and Theorem 2.3, we have the following:

Corollary 5.3. *Every normal Reed space is collectionwise normal.*

Problem 5.4. Let $t(X) \leq \omega$. Let $X \times \omega_1$ be normal and every increasing open cover of X have a closed shrinking. Then does every open cover of size ω_1 have a closed shrinking?

Problem 5.5. Let $t(X) \leq \omega$. If every open cover of X has a closed shrinking, then is $X \times \omega_1$ normal?

Problem. 5.6. Let X be a perfectly normal collectionwise normal space with $t(X) \leq \omega$. Then is $X \times \omega_1$ normal?

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